Complex Analysis: Resit Exam

Aletta Jacobshal 03, Friday 8 April 2016, 14:00 – 17:00
Exam duration: 3 hours

Instructions — read carefully before starting

- Do not forget to very clearly write your **full name** and **student number** on each answer sheet and on the envelope. Do not seal the ennvelope.
- The exam consists of 6 questions; answer all of them.
- The total number of points is 100 and 10 points are "free". The exam grade is the total number of points divided by 10.
- Solutions should be complete and clearly present your reasoning. If you use known results (lemmas, theorems, formulas, etc.) you must explain why the conditions for using such results are satisfied.
- You are allowed to have a 2-sided A4-sized paper with handwritten notes.

Question 1 (12 points)

(a) (6 points) Verify that the function $f(z) = (z+i)^2$ satisfies the Cauchy-Riemann equations.

Solution

Write

$$f(z) = (z+i)^2 = (x + i(y+1))^2 = x^2 - (y+1)^2 + 2ix(y+1),$$

and identify

$$u = x^2 - (y+1)^2, \quad v = 2x(y+1).$$

Then we have that

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y},$$

and

$$\frac{\partial u}{\partial y} = -2(y+1) = -\frac{\partial v}{\partial x}.$$

Therefore the Cauchy-Riemann equations are satisfied.

(b) (6 points) Compute the Taylor series of the function $f(z) = (z+i)^2$ around $z_0 = 1 \in \mathbb{C}$. What is the domain where this Taylor series converges?

Solution

Write $w = z - 1$. Then

$$\begin{aligned} f(z) &= (z+i)^2 = (w+1+i)^2 = w^2 + 2(1+i)w + (1+i)^2 = w^2 + (2+2i)w + 2i \\ &= 2i + (2+2i)(z-1) + (z-1)^2. \end{aligned}$$

The domain of convergence is obviously $\mathbb{C}$ (since the series is here a finite sum).

Question 2 (18 points)

Consider the function

$$f(z) = \frac{e^{-iz}}{z^2 + 4}.$$

(a) (6 points) Compute the residue of $f(z)$ at each one of the singularities of $f(z)$.

Solution

The singularities of $f(z)$ are $z = 2i$ and $z = -2i$, obtained as solutions of $z^2 + 4 = 0$. Each of the singularities is a pole of order 1. Therefore,

$$\operatorname{Res}(f; 2i) = \lim_{z \to 2i} (z - 2i) \frac{e^{-iz}}{z^2 + 4} = \lim_{z \to 2i} \frac{e^{-iz}}{z + 2i} = \frac{e^2}{4i} = -\frac{e^2}{4} i,$$

and

$$\operatorname{Res}(f; -2i) = \lim_{z \to -2i} (z + 2i) \frac{e^{-iz}}{z^2 + 4} = \lim_{z \to -2i} \frac{e^{-iz}}{z - 2i} = -\frac{e^{-2}}{4i} = \frac{1}{4e^2} i.$$

(b) (12 points) Compute the principal value

$$\operatorname{pv} \int_{-\infty}^{\infty} \frac{e^{-ix}}{x^2 + 4} \, dx.$$

Solution

We have

$$I = \operatorname{pv} \int_{-\infty}^{\infty} \frac{e^{-ix}}{x^2 + 4} \, dx = \lim_{R \to \infty} \int_{-R}^{R} \frac{e^{-ix}}{x^2 + 4} \, dx.$$

Defining the contour γ as the straight line along the real axis from $-R$ to R, we can write

$$I = \lim_{R \to \infty} \int_{\gamma} \frac{e^{-iz}}{z^2 + 4} \, dz.$$

We define a closed negatively oriented contour Γ as

$$\Gamma = \gamma + C_R^-,$$

where C_R^- is the half-circle of radius R and center 0 in the lower half-plane joining R to $-R$.

Then

$$\int_{\Gamma} \frac{e^{-iz}}{z^2 + 4} \, dz = -2\pi i \operatorname{Res}(f; -2i) = \frac{\pi}{2e^2}$$

for large enough values of R since Γ encloses only the singularity $-2i$ of the integrand and it is negatively oriented.

For the integral over C_R^- we have that the coefficient of iz in e^{-iz} is negative, and the degree of the denominator in $1/(z^2 + 4)$ is 2 while the degree of the numerator is 0, and we can apply Jordan's lemma to get

$$\lim_{R \to \infty} \int_{C_R^-} \frac{e^{-iz}}{z^2 + 4} \, dz = 0.$$

Therefore

$$\lim_{R\to\infty}\left(\int_\gamma+\int_{C_R^-}\right)\frac{e^{-iz}}{z^2+4}\,dz=\lim_{R\to\infty}\int_\Gamma\frac{e^{-iz}}{z^2+4}\,dz=\frac{\pi}{2e^2},$$

and the left-hand side gives

$$I+0=\frac{\pi}{2e^2}.$$

From here

$$I=\frac{\pi}{2e^2}.$$

Question 3 (14 points)

Consider the branch $f(z)=\mathcal{L}_{2\pi}(z)$ of the logarithm.

(a) (6 points) Compute $f(-e)$ and $f'(-e)$. Write the results in Cartesian form.

Solution

We have

$$f(-e)=\mathcal{L}_{2\pi}(-e)=\operatorname{Log}|-e|+i\arg_{2\pi}(-e)=1+3\pi i.$$

Moreover,

$$f'(z)=\frac{1}{z},$$

so

$$f'(-e)=-\frac{1}{e}.$$

(b) (8 points) Evaluate the limits $\lim_{\varepsilon\to0^+}f(x+i\varepsilon)$ and $\lim_{\varepsilon\to0^+}f(x-i\varepsilon)$ for $x>0$.

Solution

We have

$$\begin{aligned}\lim_{\varepsilon\to0^+}f(x+i\varepsilon)&=\lim_{\varepsilon\to0^+}\mathcal{L}_{2\pi}(x+i\varepsilon)\\&=\lim_{\varepsilon\to0^+}\operatorname{Log}|x+i\varepsilon|+i\lim_{\varepsilon\to0^+}\arg_{2\pi}(x+i\varepsilon)\\&=\operatorname{Log}|x|+2\pi i=\operatorname{Log}x+2\pi i.\end{aligned}$$

We used here that the function $\operatorname{Log}|z|$ is continuous so the first limit is $\operatorname{Log}|x|$ while for $x>0$ and $\varepsilon>0$ the second limit is 2π. Then we have

$$\begin{aligned}\lim_{\varepsilon\to0^+}f(x-i\varepsilon)&=\lim_{\varepsilon\to0^+}\mathcal{L}_{2\pi}(x-i\varepsilon)\\&=\lim_{\varepsilon\to0^+}\operatorname{Log}|x-i\varepsilon|+i\lim_{\varepsilon\to0^+}\arg_{2\pi}(x-i\varepsilon)\\&=\operatorname{Log}|x|+4\pi i=\operatorname{Log}x+4\pi i.\end{aligned}$$

We used here again that the function $\operatorname{Log}|z|$ is continuous so the first limit is $\operatorname{Log}|x|$ while for $x>0$ and $\varepsilon>0$ the second limit is 4π.

Question 4 (14 points)

Consider the function

$$f(z) = \frac{z^2}{z-2}.$$

(a) (4 points) Determine the singularities of $f(z)$ and their type.

Solution

The function has only one singularity at $z = 2$ and it is a simple pole (pole of order 1).

(b) (10 points) Compute the Laurent series $\sum_{j=-\infty}^{\infty} a_j z^j$ of the function $f(z)$ in the domain $|z| > 2$. What is the value of a_{-1}?

Solution

We have

$$\frac{z^2}{z-2} = \frac{z}{1-\frac{2}{z}} = z\sum_{j=0}^{\infty}\frac{2^j}{z^j} = \sum_{j=0}^{\infty}\frac{2^j}{z^{j-1}} = \sum_{j=-1}^{\infty}\frac{2^{j+1}}{z^j},$$

where we used the geometric series since $|2/z| < 1$. To write the last expression in the standard form we let $j \to -j$ and we find

$$\frac{z^2}{z-2} = \sum_{j=-\infty}^{1} 2^{1-j} z^j.$$

Then $a_{-1} = 2^{1-(-1)} = 4$.

Question 5 (16 points)

(a) (6 points) Given the function

$$f(z) = \frac{z^3\,(z-3i)\,(z+1)^2}{z^2+2i},$$

evaluate the integral

$$\int_C \frac{f'(z)}{f(z)}\,dz,$$

where C is the positively oriented circular contour with $|z| = 2$.

Solution

The Argument Principle gives that under the assumptions in this question we have

$$\int_C \frac{f'(z)}{f(z)}\,dz = 2\pi i[N_0(f) - N_p(f)],$$

where $N_0(f)$ is the number of zeros of $f(z)$ inside C, counting multiplicities, and $N_p(f)$ is the number of poles of $f(z)$ inside C, counting orders.

The function $f(z)$ has zeros at 0 (triple zero), -1 (double zero), and $3i$. The only zeros inside C are -1 (double) and 0 (triple). Therefore, $N_0(f) = 5$. The poles are solutions of $z^2 + 2i$, so there are two poles and they both lie on the circle $|z| = \sqrt{2}$, that is, inside C. Therefore,

$$\int_C \frac{f'(z)}{f(z)}\,dz = 2\pi i[5-2] = 6\pi i.$$

(b) (10 points) Use Rouché's theorem to show that the polynomial $P(z) = z^3 + \varepsilon(z^2 + 1)$, where $0 < \varepsilon < 8/5$, has exactly 3 roots in the disk $|z| < 2$.

Solution

We apply Rouché's theorem with $f(z) = z^3$ and $h(z) = \varepsilon(z^2 + 1)$. The function $f(z)$ has exactly 3 zeros (counting multiplicity) and they lie in the disk $|z| < 2$. To conclude that $P(z)$ also has exactly two zeros inside the same disk we must check that $|h(z)| < |f(z)|$ on the circle $|z| = 2$.

For $|z| = 2$ and $\varepsilon > 0$ we have

$$|f(z)| = |z^3| = |z|^3 = 8,$$

and

$$|h(z)| = |\varepsilon||z^2 + 1| \leq |\varepsilon|(|z^2| + 1) = 5\varepsilon.$$

Therefore, for $0 < \varepsilon < 8/5$ and for $|z| = 2$ we have $|h(z)| = 5\varepsilon < 8 = |f(z)|$. Applying Rouché's theorem gives the required result.

Question 6 (16 points)

(a) (8 points) Show that

$$\left| \int_C \frac{e^z}{\bar{z} + 2}\, dz \right| \leq \pi e^2,$$

where C is the positively oriented circle $|z - 1| = 1$.

Solution

On C we have that $0 \leq x \leq 2$ where $x = \operatorname{Re} z$. It is possible to see this by drawing C or by noticing that $x - 1 = \operatorname{Re}(z) - 1 = \operatorname{Re}(z - 1)$ and we always have $|\operatorname{Re} w| \leq |w|$, so $|x - 1| \leq 1$. Therefore,

$$|e^z| = |e^x e^{iy}| = e^x \leq e^2.$$

Moreover,

$$|\bar{z} + 2| = |(\bar{z} - 1) + 3| \geq ||\bar{z} - 1| - 3| = |1 - 3| = 2.$$

Therefore,

$$\left| \frac{e^z}{\bar{z} + 2} \right| \leq \frac{e^2}{2}.$$

This means

$$\left| \int_C \frac{e^z}{\bar{z} + 2}\, dz \right| \leq \frac{e^2}{2} \ell(C) = \pi e^2,$$

where, in the last step, we used that the length of the circle C of radius 1 is 2π.

(b) (8 points) Suppose that $f(z)$ is an entire function such that $f(z)/z^2$ is bounded for $|z| \geq R$, where $R > 0$. Show that $f(z)$ is a polynomial of degree at most 2.

Solution

Since $f(z)$ is entire its Laurent series around any $z_0 \in \mathbb{C}$ coincides with its Taylor series and converges everywhere. In particular, the Laurent series for $|z| \geq R$ is the Taylor series at 0 and it is given by

$$f(z) = \sum_{j=0}^{\infty} a_j z^j.$$

The function $f(z)/z^2$ is only singular at $z = 0$, therefore its Laurent series is given for $|z| \geq R$ by

$$\frac{f(z)}{z^2} = \frac{a_0}{z^2} + \frac{a_1}{z} + \sum_{j=0}^{\infty} a_{j+2} z^j.$$

The last series defines an entire function and it is bounded since for $|z| \geq R$ we have

$$\left|\frac{f(z)}{z^2} - \frac{a_0}{z^2} - \frac{a_1}{z}\right| \leq \left|\frac{f(z)}{z^2}\right| + \left|\frac{a_0}{z^2}\right| + \left|\frac{a_1}{z}\right| \leq M + |a_0|R^{-2} + |a_1|R^{-1} = M'.$$

Therefore $\sum_{j=0}^{\infty} a_{j+2} z^j$ is constant and taking $z = 0$ we find that it is equal to a_2. This means

$$\frac{f(z)}{z^2} = \frac{a_0}{z^2} + \frac{a_1}{z} + a_2,$$

and then

$$f(z) = a_0 + a_1 z + a_2 z^2.$$

End of the exam (Total: 90 points)